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Pythagorean M-Fuzzy subgroups under a T-norm

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Abstract:
This study explores the structural properties of Pythagorean M-fuzzy groups, building upon prior works by several researchers. It introduces the Pythagorean M-Fuzzy subgroup as a generalization of intuitionistic fuzzy subgroups, along with Pythagorean M-Fuzzy cosets and Pythagorean M-Fuzz normal subgroups. The study also discusses the impact of group homomorphisms on Pythagorean M- Fuzzy subgroups under a T-norm.

Keywords: Pythagorean fuzzy set, Pythagorean fuzzy subgroup, M-fuzzy subgroup, Pythagorean M-fuzzy subgroup.

مجموعات فيثاغورس الضبابية-M الجزئية تحت قاعدةT

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الملخص:
تستكشف هذه الدراسة الخصائص البنيوية للمجموعات الضبابيةM – فيثاغورس، بناءً على أعمال سابقة للعديد من الباحثين. وهي تقدم المجموعة الجزئية الضبابية M – فيثاغورس كتعميم للمجموعات الضبابية البديهية، جنبا الى جنب مع مجموعات فيثاغورس

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M -. الضبابية و المجموعات الجزئية الضبابية M. الطبيعية فيثاغورس. تناقش الدراسة أيضًا تأثير تشابهات المجموعات على المجموعات الجزئية الضبابية M– فيثاغورس تحت معيار T .

الكلمات المفتاحية: مجموعة فيثاغورس الضبابية، مجموعة فيثاغورس الضبابية الجزئية، المجموعة الجزئية الضبابية M ومجموعة فيثاغورس الضبابية الجزئية M.

Introduction:

Fuzzy groups are an important topic in abstract algebra and have garnered significant attention from scientists and researchers due to their practical and theoretical implications. The concept of fuzzy sets was introduced by (Lotfi Zadeh, 1965), which laid the foundation for extensive research into the generalizations of fuzzy set theory. Following this (K. Atanassov, 1983), explored Intuitionistic fuzzy sets, expanding the fuzzy logic framework.

The study of fuzzy groups began with (Rosenfeld, 1971), who introduced fuzzy groups that operate on fuzzy sets. Subsequent research included the examination of fuzzy normal subgroups, with (W.X. Gu, 1994) advancing the idea of fuzzy groups with operators. Additionally, (Jianming Zhan et al, 1984) investigated Intuitionistic Pythagorean M-fuzzy groups, further enriching the discourse in the field.

In this paper, we will establish a set of definitions and theories concerning Pythagorean M-fuzzy subgroups under a T-norm. Our focus will be on the theories and definitions outlined in section 4.

2. Preliminaries

In this section, we will recap some important definitions and concepts studied by several researchers, such as (Lotfi Zadeh, 1965), (W. X. Gu, 1994), (K. Atanassov et al 1983), (Krassimir 1986), (P. K. Sharma, 2011) and (R. R. Yager, 2013, 2016), which will be developed later.

Definition2.1:

Let G be a crisp set $\mu: G \longrightarrow [0, 1]$ is called a fuzzy subset of G. Here $\mu(x)$ is the degree of membership $x \in G$.

Definition of T-Operators2-2:

Many researchers have provided definitions of T-Operators, using T-Norm, T-Conorm, and negation functions to calculate the membership values of the intersection, union, and complement of fuzzy sets. This section, however, attempts to present a comprehensive set of definitions of T-operators.

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Definition2-3:
Let $T: [0,1] \times [0,1] \to [0,1]$. T is a T-Norm, if and only if for all $x, y, z \in [0,1]$:

1. $T\ (x, y) = T\ (y, x)$ (commutativity).
2. $T\ (x, y) \leq T\ (x, z), if\ y \leq z$ (monotonicity).
3. $T\ (x, T\ (y, z)) = T\ (T\ (x, y), z)$ (associativity).
4. $T(x, 1) = x$ if and only if $T(x, y)$ is continuous (T-Norm is Archimedean).
5. $T(x, x) < x$ for all $x \in (0,1)$ if and only if $T(x', y') < (x, y)$, $if\ x' < x, ' < y,$ for all $x', y', x, y \in (0,1)$ (strict).

Remark2-1:
Some important T-norms are $\boldsymbol{T_m}(x, y) = \min\{x, y\}$, $\boldsymbol{T_b}(x, y) = \max\{0, x + y - 1\}$$\boldsymbol{T_p(x, y) = xy}$, which are called standard intersection, bounded sum and algebraic product respectively.

Lemma2-1:
Let T be a T-norm. Then $T(T(x, y), T(w, z)) = T(T(x, w), T\ (y, z))$, for all $x, y,\ w \in [0,1]$.

Definition2-4:
The intersection of fuzzy subsets μ_1, μ_2 in a set X concerning a T-Norm is defined as the fuzzy subset $\mu = \mu 1 \cap \mu 2$ in the set X such that for any $\boldsymbol{x} \in \boldsymbol{X}$:
$\mu(x) = (\mu_1 \cap \mu_2)(x) = T\ (\mu_1(x), \mu_2(x))$.

Definition2-5:
Let $T^*: [0,1] \times [0,1] \to [0,1]$. $T^*$is a T-Conorm, If and only if for all $x, y, z \in [0,1]$:

1. $T^*(x, y) = T^*(y, x)$
2. $T^*(x, y) \leq T^*(x, z),\ if\ y \leq z$
3. $T^*(x, T^*(y, z)) = T^*(T^*(x, y), z)$
4. $T^*(x, 0) = x$ if and only if T^* is continuous (T-Conorm is Archimedean)
5. $T^*(x, x) > x\ \ \forall x \in (0,1)$ if and only if $T^*(x', y') < T^*(x, y)$, $if\ x' < x, ' < y,\ \ \forall x', y', x, y \in (0,1)$ (a strict).

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Remark2-2:

From the above definitions, we note that for a T-norm T and a T-Conorm T^*:

1- $T(0,0)=0, \quad T(1,1)=1.$

2- $T^*(0,0)=0, \quad T^*(1,1)=1.$

Definition2-6:

Let $\mu: G \rightarrow [0,1]$ be a fuzzy subset of a group $(G, \circ)$. Then μ is said to be a fuzzy subgroup of $(G, \circ)$ if the following conditions hold:

1. $\mu(x \circ y) \geq \mu(x) \cdot \mu(y)$ for all $x, y \in G$.
2. $\mu(x^{-1}) \geq \mu(x)$ for all $x \in G$.

Definition2-7:

Let μ be a fuzzy set in X. The weak *t-cut* of μ denoted by μ_t, is defined as:

$$\mu_t = \{x \in X: (x) \geq t, \ t \in [0,1]\}.$$

Definition2-8:

For any $t \in [0,1]$ and fuzzy set μ of X, the set $(\mu, t) = \{x \in X: \mu(x) \geq t\}$ (respectively, $(\mu, t) = \{x \in X: \mu(x) \leq t\}$), is called an upper (respectively, lower) t- level cut of μ. Notice that $(\mu, t) = \mu_t$.

Definition2-9:

Let G be a group. An intuitionistic fuzzy group (IFG) I on G, defined by $I = \{(x, \mu(x), v(x)) : x \in G\}$ where $\mu(x) \in [0,1]$ and $v(x) \in [0,1]$ are the degrees of membership and non -membership of $x \in G$ respectively. These satisfy the condition

$$0 \leq \mu(x) + v(x) \leq 1 \text{ for all } x \in G.$$

Definition2-10: Intuitionistic Fuzzy Subgroup (IFSG)

Let $I = \{(x, \mu(x), v(x)) : x \in G\}$ be an intuitionistic fuzzy set (IFS) *of* a group (G, o). Then I is said to be an intuitionistic fuzzy subgroup *(IFSG)* of G if the following conditions are satisfied:

1. For all $x, y \in G$:

- $\mu(x \circ y) \geq \mu(x) \wedge \mu(y)$
- $v(x \circ y) \leq v(x) \vee v(y)$

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2. For all $x \in G$:
   - $\mu(x^{-1}) \geq \mu(x)$
   - $v(x^{-1}) \leq v(x)$

Definition2-11: Pythagorean Fuzzy Set (PFS)
Let G be a crisp set. A Pythagorean fuzzy set (PFS) $\boldsymbol{A}$ on G is defined as:
$A = \{(x, \mu(x), v(x) : x \in G\}$ where $\mu(x)$ and $v(x)$ are the degrees of membership and non-membership of $x \in G$ respectively, satisfying: $0 \leq \mu^2(x) + v^2(x) \leq 1$ $\forall x \in G$.
Definition2-12:
Let $A_1 = \{(x, \mu_1(x), v_1(x)) : x \in G\}$ and $A_2 = \{(x, \mu_2(x), v_2(x)) : x \in G\}$ be two PFS of G. Then the following operations hold:

- $T^*(A_1, A_2) = \{(x, T^*(\mu_1(x), \mu_2(x))), T(v_1(x), v_2(x))) : x \in G\}$.
- $T(A_1, A_2) = \{(x, T(\mu_1(x), \mu_2(x)), T^*(v_1(x), v_2(x)) : x \in G\}$.
- $A_1^c = \{(x, v_1(x), \mu_1(x)) : x \in G\}$.
- $A_1 \subseteq A_2$ if $\mu_1(x) \leq \mu_2(x)$ and $v_1(x) \geq v_2(x)$ for all $x \in G$.
- $A_1 = A_2$ if $\mu_1(x) = \mu_2(x)$ and $v_1(x) = v_2(x)$ for all $x \in G$.

Throughout this paper, we will refer to a Pythagorean fuzzy subset as PFS and denote $A = (\mu, v)$ instead of $A = \{(x, \mu(x), v(x)) : x \in G\}$.

3. Pythagorean fuzzy subgroup

Many scientists have studied the Pythagorean, one of the most important (S. Bhunie G. Ghorai et al, 2021), in this regard; we will define the Pythagorean fuzzy subgroup (PFSG) as an extension of the fuzzy subgroup and the intuitionistic fuzzy subgroup (IFSG).

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Definition3-1:
Let $(G, \circ)$ be a group and $A = (\mu, v)$ be a PFS (Pythagorean fuzzy set) of G. The set A is classified as a PFSG (Pythagorean fuzzy subgroup) of G if the following conditions are satisfied:

1. $\mu^2 (x \circ y) \geq T(\mu^2(x), \mu^2(y))$, for all $x, y \in G$.
2. $v^2 (x \circ y) \leq T^*(v^2(x), v^2(y))$, for all $x, y \in G$.
3. $\mu^2 (x^{-1}) \geq \mu^2(x)$, for all $x \in G$.
4. $v^2 (x^{-1}) \leq v^2(x)$, for all $x \in G$.
5. $\mu^2 (x) = \{\mu(x)\}^2$ and $v^2(x) = \{v(x)\}^2$ for all $x \in G$.

Example3-1:
Consider the set $G = \{1, -1, i, -i\}$. The operation $(G, \cdot)$ forms a group under standard multiplication.

Define the PFS $A = (\mu, v)$ on G as follows:

- $\mu(1) = 0.7$
- $\mu(-1) = 0.4$
- $\mu(i) = 0.3$
- $\mu(-i) = 0.3$,
- $v(1) = 0.2$
- $v(-1) = 0.1$,
- $v(i) = 0.1$
- $v(-i) = 0.1$.

Calculating values, we found:

- $\mu^2(i, -i) = \mu^2(1) = (0.7)^2 = 0.49$
- $v^2(i, -i) = v^2(1) = (0.2)^2 = 0.04$.

Then:

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- $\mu^2(i) \cdot \mu^2(-i)$ =min{0,09, 0.09} = 0.09
- $v^2(i) \cdot v^2(-i)$ =max {0.01, 0.01} = 0.01.

Hence, we have:

- $\mu^2(i, -i) > T(\mu^2(i), \mu^2(-i))$
- $v^2(i, -i) < T^*(v^2(i), v^2(-i))$.

It can be shown similarly that $\mu^2(x \cdot y) \geq T(\mu^2(x), \mu^2(y))$ and

$v^2(x.y) \leq T^*\left(v^2(x), v^2(y)\right) for\ all\ x, y \in G.$ Furthermore, $\mu^2(x^{-1}) \geq \mu^2(x)$ and

$v^2(x^{-1}) \leq v^2(x) for\ all\ x \in G.$ Thus, A is a PFSG of $(G,\cdot)$.

Proposition3-1:
Let $A = (\mu, v)$ be a PFSG of a group $(G, \circ)$, then the following statements are true:

1. For all $x \in G$:
   - $\mu^2(e) \geq \mu^2(x)$
   - $v^2(e) \leq v^2(x)$
2. For all $x \in G$:
   - $\mu^2(x^{-1}) = \mu^2(x)$
   - $v^2(x^{-1}) = v^2(x)$

Where e is the identity element in G.

Theorem3-1:
If $A = (\mu, v)$ is an IFSG of a group (G, ○), then A is also a PFSG of the group $(G, \circ)$.

Example3-2:
Consider the Klein's 4-group $G = \{e, a, b, c\}$, where

- $a^2 = b^2 = c^2 = e$
- $ab = c, bc = a, ca = b.$

Define a Pythagorean fuzzy structure $A = (\mu, v)$ on G with the following values:

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- $\mu(e) = 0.9$
- $\mu(c) = 0.8$,
- $\mu(a) = 0.6$
- $\mu(b) = 0.6$
- $v(e) = 0.2$
- $v(c) = 0.3$
- $v(a) = 0.4$
- $v(b) = 0.4$

That is a PFSG of G.

However, $\mu(e) + v(e) = 1.1$, which exceeds 1. Thus, A is not an (IFS) of G and consequently, not an (IFSG). This example illustrates that a PFSG may not necessarily be an IFSG.

Remark3-1:
Every IFSG of a group $(G, \circ)$ is a PFSG of $(G, \circ)$, but the reverse is not necessarily true.

Definition3-2:
Let $A = (\mu, v)$ be a PFSG of a group $(G, \circ)$. The Pythagorean fuzzy left coset of A is denoted as $xA = (x\mu, xv)$, defined by:

- $(x\mu)^2(y) = \mu^2(x^{-1} \circ y)$
- $(xv)^2(y) = v^2(x^{-1} \circ y)$

The Pythagorean fuzzy right coset of A is denoted as $Ax = (\mu x, vx)$ defined by:

- $(\mu x)^2(y) = \mu^2(y \circ x^{-1})$
- $(vx)^2(y) = v^2(y \circ x^{-1})$ for all $y \in G$.

Definition3-3:
Let $A = (\mu, v)$ be a PFSG of a group $(G, \circ)$. A is a Pythagorean fuzzy normal subgroup (PFNSG) of $(G, \circ)$ if every Pythagorean fuzzy left coset of A is also a Pythagorean fuzzy right coset of A in G.
This is equivalent to stating that $xA = Ax$ for all $x \in G$.

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Definition3-4:

Let G be a crisp set. A Pythagorean fuzzy set (PFS) A = (μ, v) be a PFS of the set G is defined such that for t_1 and t_2 ∈ [0, 1], the set (t_1, t_2) = $\{x \in G: (x) \geq t_1 and\ (x) \leq t_2\}$ is called a Pythagorean fuzzy level subset (PFLS) of the PFS A, where

$$0 \leq t_1^2 + t_2^2 \leq 1.$$

Definition3-5:

The subgroup (t_1, t_2) of the group (G, ○) is referred to as a Pythagorean fuzzy level subgroup (PFLSG) of the PFSG group A = (μ, v).

4. Pythagorean M-Fuzzy subgroup

This section is the focus of our study. We will examine the Pythagorean M-Fuzzy subgroup, along with some theories and introductory definitions relevant to our research
(Jianming Zhan et al, 2004), (Muthuraj, R. et al, 2010) and
(S. Bhunie G. Ghorai et al, 2021).

Definition4-1:

An algebraic system called an M-Group consists of a group G, a set M and a function defined in the product set $M \times G$ that maps to G. For an element $m \in M$ and $x, y \in G$, if $m(x)$ denotes the element in G determined by m and x. The condition $m(xy)$ = $m(x)\, m(y)$ holds for all $x, y \in G$ and $m \in M$.

Definition4-2:

A subgroup A of an M-group G is called an M- subgroup if $\mu(x) \in A$ for all $m \in M$ and $x \in A$.

Example4-1:

Consider the group (G, o) = (E, +) where =$\{0, \pm2, \pm4, \pm6, \cdots\}$. Let $M = \{3, 5\}$

Define a function : $M \times E \longrightarrow E$ by:

- $f(m, a) = m{\cdot}a$ for all $m \in M, a \in A$
- $f(m,\ a + b) = f(m,\ a) + f(m, b)$ for all $m \in M, a, b \in E$.
  Then (G, o) is an M –group.

Let $A = \{0, \pm4, \pm8, \pm12, \pm16 \ldots\}$; thus, A is an M-Subgroup.

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Definition4-3:

Let G and $G^{`}$ be $M$- groups, and let $f$ *be* a homeomorphism from $G$ onto $G^{`}$. If $f\ (m(x)) = m\ (f(x))$ for all $m \in M$ and $x \in G$, then f is called an M- homomorphism.

Definition4-4:

Let G be an M-group and μ be a fuzzy subgroup of G if the condition $\mu(mx) \geq (x)$ holds for any $x \in G$ and $m \in M$, then μ is called an M-fuzzy subgroup of G.

Definition4-5:

Let G be an M-group. If it is both an M- fuzzy subgroup of G and a normal fuzzy subgroup of G, then μ is referred to as an M-normal fuzzy subgroup of G.

Theorem4-1:

Let G be an M-group and let μ and υ be M-fuzzy subgroups of G. Then the pair (μ, υ) forms an M-fuzzy subgroup of G.

Theorem4-2:

If μ is an M-fuzzy subgroup of an M-group G, the following statements hold for all $x, y \in G, m \in M$.

1. $\mu(m(xy)) \geq min\{\mu(x), \mu(y)\}$
2. $\mu(mx^{-1}) \geq \mu(x)$

Theorem4-3:

Let G be an M-group and μ be a fuzzy set of G. Then μ is the M-fuzzy subgroup of G if and only if for all $t \in [0, 1]$, μ_t is an M-subgroup of G, whenever $\mu_t \neq \emptyset$

Corollary4-1:

Let μ be a fuzzy set of the M-group G. Then μ is an M-normal fuzzy subgroup if and only if μ_t is an M-normal subgroup of G for any $t \in [0, 1]$, where $\mu_t \neq \emptyset$.

Theorem4-4:

Let G_1 *and* G_2 be an M-group and let and υ be two M-fuzzy subgroups of G_2. If it is an M-normal fuzzy subgroup of υ and is an M-homomorphism from G_1 *to* G_2, then $f^{-1}\ (\mu)$ is an M-normal fuzzy subgroup of $f^{-1}\ (\upsilon)$.

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Theorem4-5:
Let f be an M-homomorphism from the M-group G_1 to the M-group G_2. Then:

1. If μ is an M-fuzzy subgroup of G_2, then $f^{-1}(\mu)$ is an M-fuzzy subgroup of G_1.
2. If μ is an M-normal fuzzy subgroup of G_2, then $f^{-1}(\mu)$ is an M-normal fuzzy subgroup of G_1.

Definition4-6:
Let G be an M-group and let PFS $A = (\mu, v)$ be a Pythagorean fuzzy subgroup of G.

If $\mu^2(mx) \geq \mu^2(x)$ and $v^2(mx) \leq v^2(x)$ for all $x \in G, m \in M$, then PFS $A =$

(μ, v) is said to be a Pythagorean fuzzy subgroup with operators of G. We will use the phrase" $A = (\mu, v)$ is a Pythagorean M-fuzzy subgroup of G" instead of "a Pythagorean fuzzy subgroup with operators of G".

Theorem4-6:
Let (G_1, o_1) and (G_2, o_2) be two groups. If f is a surjective homomorphism from (G_1, o_1) *to* (G_2, o_2) *and* $A = (\mu, v)$ is a PFSG of(G_1, o_1). Then $A' = (f(\mu), f(v))$ is a PFSG of (G_2, o_2).

Proof:
Since $f: G_1 \to G_2$ is a surjective homomorphism, then $f(G_1) = G_2$
Let x_2 *and* y_2 be two elements of G_2.
Suppose $x_2 = f(x_1)$ and $y_2 = f(y_1)$ for some $x_1, y_1 \in G_1$.
We have$A' = \{(b, f(\mu)(b), f(v)(b)): b \in G_2\}$.
Now:

$$
\begin{aligned}
\big(f(\mu)\big)^2(x_2 \circ_2 y_2) &= \{(f(\mu))(x_2 \circ_2 y_2)\}^2 \\
&= [T^*\{\mu(a): a \in G_1, f(a) = x_2 \circ_2 y_2\}]^2 \\
&= T^*\{\mu^2(a): a \in G_1, f(a) = x_2 \circ_2 y_2\} \\
&\geq T^*\{\mu^2(x_1 \circ_1 y_1): x_1, y_1 \in G_1 \textit{ and } f(x_1) = x_2, f(y_1) = y_2\} \\
&\geq T^*\{\mu^2(x_1) \wedge \mu^2(y_1): x_1, y_1 \in G_1 \textit{ and } f(x_1) = x_2, f(y_1) = y_2\} \\
&= (T^*\{\mu^2(x_1): x_1 \in G_1 \textit{ and } f(x_1) = x_2\}) T (T^*\{\mu^2(y_1): y_1 \in G_1 \textit{ and } f(y_1) = y_2\}) \\
&= \{(f(\mu))(x_2)\}^2 T \{(f(\mu))(y_2)\}^2 \\
&= \big(f(\mu)\big)^2(x_2) T \big(f(\mu)\big)^2(y_2).
\end{aligned}
$$

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Therefore $\left(f(\mu)\right)^2(x_2 \circ_2 y_2) \geq \left(f(\mu)\right)^2(x_2)T\left(f(\mu)\right)^2(y_2)$ for all x_2 and $y_2 \in G_2$.

Similarly, we can prove that $\left(f(v)\right)^2(x_2 \circ_2 y_2) \leq \left(f(v)\right)^2(x_2)T^*\left(f(v)\right)^2(y_2)$ for all x_2 and $y_2 \in G_2$.

Also, we have

$$\left(f(\mu)\right)^2(x_2^{-1}) = \{f(\mu)(x_2^{-1})\}^2$$
$$= [T^*\{\mu(x): x \in G_1 \; and \; f(x) = (x_2^{-1})\}]^2$$
$$= [T^*\{\mu(x^{-1}): x^{-1} \in G_1 \; and \; f(x^{-1}) = x_2\}]^2$$
$$= \{f(\mu)(x_2)\}^2$$

$= \left(f(\mu)\right)^2(x_2)$.

Therefore $\left(f(\mu)\right)^2(x_2^{-1}) = \left(f(\mu)\right)^2(x_2)$ for all $x_2 \in G_2$.

Similarly, we can show that $\left(f(v)\right)^2(x_2^{-1}) = \left(f^{-1}(v)\right)^2(x_2)$ for all $x_2 \in G_2$.

Hence $A' = (f(\mu), f(v))$ is PFSG of $(G_2, \circ_2)$.

Proposition4-1:

If PFS $A = (\mu, v)$ is a Pythagorean M-fuzzy subgroup of an M-group G, then for any $x, y \in G \; and \; m \in M$.

1. $\mu^2(m(xy)) \geq \mu^2(mx) \cdot \mu^2(my)$ and $v^2(m(xy)) \leq v^2(mx) \cdot v^2(my)$
2. $\mu^2(mx^{-1}) \geq \mu^2(x)$ and $v^2(mx^{-1}) \leq v^2(x)$.

Theorem4-7:

Let A be an M-subgroup of an M-group G and let PFS $A = (\mu, v)$ be a Pythagorean fuzzy set in G defined by:

$$\mu(x) = \begin{matrix} t_1 & if \; x \in A \\ t_2 & otherwise \end{matrix}$$

$$v(x) = \begin{matrix} s_1 & if \; x \in A \\ s_2 & otherwise \end{matrix}$$

for all $x \in A$, where $t_1 > t_2$ and $s_1 < s_2$ in$[0, 1]$. Then PFS $A = (\mu, v)$ is a Pythagorean M-fuzzy subgroup of G.

Proof:

Assume that A is an M-subgroup of G. Let $x, y \in G$. If x and y are in A, then:

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$\mu^2(x) = t_1 = \mu^2(y)$ *and* thus $xy \in A$. Hence $\mu^2(xy) = t_1 = (\mu^2(x), \mu^2(y))$.

If either $x \notin A$ *or* $y \notin A$, then either $\mu^2(x) = t_2$ *or* $\mu^2(y) = t_2$. It follows that $\mu^2(xy) \geq t_2 \geq T(\mu^2(x), \mu^2(y))$. Similarly, we have
$v^2(xy) \leq T^*(v^2(x), v^2(y))$. Now, let $x \in G$. If $x \in A$, then $\mu^2(x) = t_1$ *and* $x^{-1} \in$

A. Thus $\mu^2(x) = t_1 = \mu^2(x^{-1})$. If $x \notin A$, then $\mu^2(x) = t_2$ and $x^{-1} \notin A$. Thus

$\mu^2(x^{-1}) = t_2 = \mu^2(x)$.

Similarly, we have $v^2(x^{-1}) = v^2(x)$. Hence PFS $A = (\mu, v)$ is a Pythagorean fuzzy subgroup of G. Since A is an M-subgroup of G, we have $mx \in A$ for all
$x \in A$ and $m \in M$..

Therefore $\mu^2(mx) = t_1 = \mu^2(x)$ and $v^2(mx) = s_1 = v^2(x)$. If

$x \notin A$, *then* $\mu^2(mx) \geq t_2 = \mu^2(x)$ *and* $v^2(mx) \leq s_2 = v^2(x)$. Consequently, PFS

$A = (\mu, v)$ is a Pythagorean M-fuzzy subgroup of G.

Theorem4-8:

Let PFS $A = (\mu_A, v_A)$ be a Pythagorean fuzzy subgroup (PFS) in an M-group G. Then PFS $A = (\mu_A, v_A)$ is a Pythagorean M-fuzzy subgroup of G if and only if the nonempty sets $(\mu_A; \mu)$ and $L(v_A; v)$ are M-subgroups of G for every $\mu \in T(Im\mu_A, Imv_A)$.

Proof

Forward Direction:

1. Let $\mu \in (Im\mu_A,\ Imv_A) \subseteq [0,1]$ and $x, y \in U(\mu_A; \mu)$
2. By definition, $\mu^2(x) \geq \mu^2 and\ \mu^2(y) \geq \mu^2$.
3. Therefore, we have: $\mu^2(xy) \geq T\big(\mu^2(x) \cdot \mu^2(y)\big) \geq \mu^2$ which implies $x, y \in U(\mu_A;\ \mu)$.
4. If $x \in (\mu_A;\ \mu)$, then $\mu^2(x) \geq \mu^2$ implies $\mu^2(x^{-1}) \geq \mu^2(x) \geq \mu^2$.
5. Thus, $x^{-1} \in (\mu_A;\ \mu)$, showing that $(\mu_A;\ \mu)$ is a subgroup of G.

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6. Now, let$x, y \in (v_A;\ \mu)$. Then $v^2(x) \le \mu^2$ and $v^2(y) \le \mu^2$:
   $v^2(xy) \le T^*(v^2(x), v^2(y)) \le \mu^2$, so $xy \in L(v_A;\ \mu)$
7. If $x \in (v_A, \mu)$ then $v^2(x) \le \mu^2$ implies $v^2(x^{-1}) \le v^2(x) \le \mu^2$
8. Hence,$x^{-1} \in L(v_A;\ \mu)$ confirms that $(v_A; \mu)$ is a subgroup of G.
9. For any $x \in (\mu_A;\ \mu)$, and $m \in M$: $\mu^2(mx) \ge \mu^2(x) \ge \mu^2$, which implies $mx \in U(\mu_A;\ \mu)$.
10. Similarly, for $x \in (v_A;\ \mu)$ and $m \in M$: $v^2(mx) \le v^2(x) \le \mu^2$, leading to $mx \in L(v_A;\ \mu)$. Thus $(\mu_A;\ \mu)$ and $L(v_A;\ \mu)$ are M-subgroups of G.

Reverse Direction:

Assume that the nonempty sets $(\mu_A;\ \mu)$ and $L(v_A;\ \mu)$ are M-subgroups of G.

1. Suppose there exist $x_0, y_0 \in G$ such that $\mu^2(x_0 y_0) < T\big(\mu^2(x_0), \mu^2(y_0)\big)$.
2. Taking: $\mu^2 = (\mu^2(x_0 y_0) + T\big(\mu^2(x_0), \mu^2(y_0)\big))\ /\ 2$, leads to

$\mu^2(x_0 y_0) < \mu^2 < T\big(\mu^2(x_0), \mu^2(y_0)\big)$.

3. This implies $x_0, y_0 \in (\mu_A;\ \mu_0)$, but

$x_0 y_0 \notin (\mu_A;\ \mu_0)$, leading to a contradiction.

4. If there exists $x_0 \in G$ such that $\mu^2(x^{-1}) < x^2(x_0)$:

$\mu^2 = \frac{\mu^2(x^{-1}) + \mu^2(x_0)}{2}$, gives $\mu^2(x^{-1}) < \mu^2 < \mu^2(x_0)$, leading to $x_0 \in (\mu_A;\ \mu_0)$, which contradicts $x^{-1} \notin U(\mu_A;\ \mu_0)$.

Thus, $\mu^2(mx) \ge \mu^2(x)\ for\ all\ x \in G\ and\ m \in M$ similarly, $v^2(mx) \le v^2(x)$ can be shown.

Therefore, PFS $A = (\mu_A, v_A)$ is a Pythagorean M-fuzzy subgroup of G.

Definition4-7:

Let $f\colon G \to G'$ be a homomorphism of M-groups. For any Pythagorean fuzzy set (PFS) $A = (\mu_A, v_A)$ in G', we define a new PFS $A_f = (\mu_f, v_f)$ in G as follows:

- $\mu_f(x) = \mu_A(f(x))$
- $v_f(x) = v_A\big(f(x)\big)\ for\ all\ x \in G$

Lemma4-1:

Let G and G' be M-groups and f a homomorphism from G onto G'.

1. If PFS $A = (\mu_A, v_A)$ is a Pythagorean fuzzy

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subgroup of G', then PFS $A_f = (\mu_f, v_f)$ is a Pythagorean fuzzy subgroup of G.

2. If PFS $A_f = (\mu_f, v_f)$ is a Pythagorean fuzzy subgroup of G, then PFS $A = (\mu_A, v_A)$ is a Pythagorean fuzzy subgroup of G'.

Theorem4-9:

Let G and G' be M-groups and f an M-homomorphism from G onto G'. If PFS $A = (\mu_A, v_A)$ is a Pythagorean M-fuzzy subgroup of G', then PFS $A_f = (\mu_f, v_f)$ is a Pythagorean M-fuzzy subgroup of G.

Proof:

By lemma 4-1, PFS $\boldsymbol{A_f} = (\boldsymbol{\mu_f}, \boldsymbol{v_f})$ is Pythagorean fuzzy subgroup of G. For any $\boldsymbol{x} \in \boldsymbol{G}$ and $\boldsymbol{m} \in \boldsymbol{M}$:

- $\boldsymbol{\mu_f(mx) = \mu_A(f(mx)) = \mu_A(mf(x)) \geq \mu_A(f(x)) = \mu_f(x)}$
- $\boldsymbol{v_f(mx) = v_A(f(mx)) = v_A(mf(x)) \leq v_A(f(x)) = v_f(x)}$

Thus, PFS $\boldsymbol{A_f} = (\boldsymbol{\mu_f}, \boldsymbol{v_f})$ is a Pythagorean M-fuzzy subgroup of G.

Theorem4-10:

Let G and G' be M-groups and f an M-homomorphism from G onto G'. If PFS $A_f = (\mu_f, v_f)$ is a Pythagorean M-fuzzy subgroup of G, then PFS $A = (\mu_A, v_A)$ is a Pythagorean M-fuzzy subgroup of G'.

Proof:

By Lemma, PFS $A = (\mu_A, v_A)$ is a Pythagorean fuzzy subgroup of G', for any $y \in G'$ and $m \in M$, $x \in f^{-1}(y)$.

For the membership function:

$\mu_f(my) = sup_{z \in f^{-1}(my)} \mu_A(z)$

$\geq sup_{mx \in f^{-1}(my)} (mx) = su\ (mx)$

$= my\ \mu_A(mx)$

$= sup_{mf(x)}$

$= my\ \mu_A(mx)$

$\geq sup_{f(x)=y} \mu_A(x) = \mu_f(y)$

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Also, we have

$v_f(my) = inf_{z \in f^{-1}(my)} v_A(z)$

$\leq inf_{mx \in f^{-1}(my)} v_A(mx) = \ inf_{f(mx)}$

$= my^{v_A}(mx) = \ inf_{mf(x)}$

$= my\ v_A(mx) \leq inf_{f(x)=y} v_A(x) = v_f(y)$

Hence, PFS $A = (\mu_A, v_A)$ is a Pythagorean M-fuzzy subgroup of G.

Conclusion:

In this study, we investigated Pythagorean M-Fuzzy subgroups by connecting M-Fuzzy subgroups with Pythagorean fuzzy subgroups using T-Norm. Our analysis primarily focused on Section 4, where we provided a set of definitions and theories. We established that an intuitionistic M-Fuzzy subgroup within any given group is inherently a Pythagorean M-Fuzzy subgroup of that group. Additionally, we introduced the concepts of Pythagorean M-Fuzzy cosets and Pythagorean M-Fuzzy normal subgroups in this context.

Furthermore, we identified the essential conditions that must be met for an M-Fuzzy subgroup to qualify as a Pythagorean M-Fuzzy normal subgroup. The study also thoroughly examined the impact of m-group homomorphism on Pythagorean M-Fuzzy subgroups.

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